

Landau Damping and Entropy

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It is shown that the well-known LANDAU-damping entropy paradox can be resolved by considering a macroscopic entropy as defined by CLAUSIUS instead of the microscopic entropy defined by BOLTZMANN. Although both entropy definitions are identical for equilibrium configurations, the same in general is not true for non-equilibrium configurations. To extend the entropy definition of CLAUSIUS to non-equilibrium configurations requires a generalization of the concept of temperature to non-equilibrium configurations. A natural generalization of the concept of temperature to non-equilibrium configurations is to put the temperature proportional to the mean square velocity fluctuation. In this way a macroscopic entropy can be defined which is then proportional to the logarithm of the mean square velocity fluctuation. In the case of LANDAU damping it can be shown that two states having the same statistical permutability and thus the same microscopic entropy may have different mean square velocity fluctuations. One should therefore consider, besides the microscopic disorder defined by BOLTZMANN, which is proportional to the logarithm of the permutability, a macroscopic disorder which is proportional to the logarithm of the mean square velocity fluctuation.

In the case of LANDAU damping the microscopic entropy does not change with time; the macroscopic entropy, however, increases steadily with time.

1. Introduction

In considering waves propagating through a collisionless plasma one can show the existence of a peculiar kind of a damping called after its discoverer LANDAU damping¹.

LANDAU found this damping for high-frequency electrostatic plasma waves. The damping mechanism, however, is not restricted to this particular kind of plasma wave, but is in fact a general property common to all kinds of plasma waves. The damping itself can be derived very elegantly from the so-called BOLTZMANN-VLASOV equation².

The case of an electronic plasma, treated by LANDAU, is distinguished by its simplicity. For this reason we will restrict the following considerations to an electronic plasma. In the case of an electronic plasma, the electrons are described by a distribution function $F(\mathbf{v}, \mathbf{r}, t)$ normalized with n being the number density as follows:

$$n = \int F d\tau; \quad (d\tau = dv_x dv_y dv_z) \quad (1.1)$$

and which in the absence of collisions must obey the collisionless BOLTZMANN equation:

$$dF/dt = 0 \quad (1.2)$$

$$\text{or} \quad \frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial F}{\partial \mathbf{v}} = 0. \quad (1.3)$$

In (1.3) \mathbf{a} is an acceleration acting on the particles. In the case of electrostatic waves the acceleration

\mathbf{a} must be expressed by the force acting on the electrons. If φ is the scalar electric potential the acceleration \mathbf{a} is then given by:

$$\mathbf{a} = -(e/m) \nabla \varphi. \quad (1.4)$$

We thus obtain for (1.3):

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{r}} - \frac{e}{m} \nabla \varphi \cdot \frac{\partial F}{\partial \mathbf{v}} = 0. \quad (1.5)$$

VLASOV has supplemented Eq. (1.5) by expressing the electrostatic potential φ through the distribution function:

$$\nabla^2 \varphi = -4\pi e \int F d\tau. \quad (1.6)$$

Eq. (1.5) together with (1.6) are called the BOLTZMANN-VLASOV equations. The equations are nonlinear due to the term $(e/m) \nabla \varphi \cdot \partial F / \partial \mathbf{v}$. To make a simplified analysis the equations have been studied in the linearized approximation describing waves of infinitesimal amplitude. Other plasma wave modes can be described by a similar set of equations. To linearize the equations one puts:

$$F(\mathbf{v}, \mathbf{r}, t) = f_0(\mathbf{v}) + f(\mathbf{v}, \mathbf{r}, t) \quad (1.7)$$

where $f_0(\mathbf{v})$ is the equilibrium distribution and $f(\mathbf{v}, \mathbf{r}, t)$ is considered small as compared with $f_0(\mathbf{v})$.

From (1.5) we obtain the linearized equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{e}{m} \nabla \varphi \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0 \quad (1.8)$$

¹ L. LANDAU, J. Phys. USSR **10**, 25 [1946].

² A. VLASOV, J. Phys. USSR **9**, 25 [1945].



and from (1.6):

$$\nabla^2 \varphi = -4\pi e \int f d\tau. \quad (1.9)$$

The equilibrium charge $e \int f_0 d\tau$ is compensated by the positive charge of the ions.

Eqs. (1.8) and (1.9) form a complete set of equations from which the LANDAU damping can be derived.

2. Microscopic and Macroscopic Entropy

In order to generalize the concept of entropy we start according to BOLTZMANN from the permutability of n particles distributed in m cells of equal size:

$$W = n! / (n_1! n_2! \dots n_m!). \quad (2.1)$$

Taking the logarithm of (2.1) and making use of STIRLING's formula, we obtain from (2.1):

$$\ln W = - \sum_{i=1}^m n_i \ln n_i + \text{const.} \quad (2.2)$$

Going from a discrete distribution in different cells over to a continuous distribution F we can write instead of (2.2):

$$\ln W = - (1/n) \int F \ln F d\tau + \text{const.} \quad (2.3)$$

If we ask for the maximum of $\ln W$ under the constraints:

$$\begin{aligned} \delta E &= \frac{1}{2} m \delta \int v^2 F d\tau = 0 && \text{conservation of energy,} \\ \delta \mathbf{p} &= m \delta \int \mathbf{v} F d\tau = 0 && \text{conservation of momentum,} \\ \delta n &= \delta \int F d\tau = 0 && \text{conservation of particle} \\ &&& \text{numbers,} \end{aligned} \quad (2.4)$$

we obtain in general a displaced MAXWELLIAN velocity distribution describing a gas in thermodynamic equilibrium moving with some constant speed. In the case of thermodynamic equilibrium we know that the entropy S is connected with $\ln W$ by:

$$S = \kappa \ln W. \quad (2.5)$$

So that if $\ln W$ has a maximum, S must have a maximum. We further know that in thermodynamic equilibrium the entropy S according to CLAUSIUS can be also expressed by:

$$\begin{aligned} S - S_0 &= \int (1/T) dQ; \\ (dQ &= c_v dT + p dV, \text{ entropy constant } S_0 \text{ arbitrary}). \end{aligned} \quad (2.6)$$

For non-equilibrium states BOLTZMANN defines the entropy by:

$$H = - (\kappa/n) \int F \ln F d\tau. \quad (2.7)$$

Comparing (2.3) with (2.5) we see that for equilibrium configurations:

$$H = S + \text{const.} \quad (2.8)$$

In BOLTZMANN's statistical concept, entropy is proportional to the logarithm of the permutability which by itself is a measure of disorder. The question now arises how can different states having the same permutability W be classified as to whether having more or less disorder.

Taking into account the constraint:

$$n = \sum_{i=1}^m n_i$$

we can conclude that W is a function of $m-1$ variables for a given cell number m . $W = \text{const}$ is therefore an equipotential curve on a $m-1$ dimensional hypersurface. Our problem is thus equivalent with the question, what is the measure of order or disorder for different points on this equipotential curve.

Speaking in terms of the distribution function F the question is equivalent to asking how to classify the disorder of two distribution functions possessing the same value H . This simple consideration indicates that for non-equilibrium distributions a second kind of measure for disorder is needed to supplement the microscopic disorder defined by BOLTZMANN according to Eq. (2.7).

If we represent the hypersurface $W = f(n_i) = \text{const}$ in a schematic way as indicated by Fig. 1, the lines

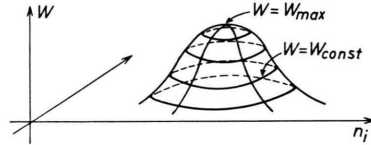


Fig. 1. To the measure of disorder in a non-equilibrium configuration.

$W = \text{const}$ are lines on which $H = \text{const}$. For this reason the second measure of disorder must have different values at different points of the equipotential curve $W = \text{const}$, resp. $H = \text{const}$. In the case of complete thermodynamic equilibrium the family of curves $W = \text{const}$ (resp. $H = \text{const}$) degenerates into just one point that is the point where $W = W_{\text{max}}$. If complete thermodynamic equilibrium is reached, the second measure of disorder can therefore acquire just one value, the value at the point where $W = W_{\text{max}}$. In thermodynamic equilibrium both measures of dis-

order should be normalized in such a way as to give identical results.

An additional measure of disorder can be obtained in a simple way from the entropy definition of CLAUSIUS given by Eq. (2.6) if the concept of temperature and pressure can be generalized to non-equilibrium states so that the heat

$$dQ = c_v dT + p dV \quad (2.9)$$

which is supplied to the system of particles and which occurs in Eq. (2.6) has a well-defined meaning.

The normalization of both measures of entropy, the definition of entropy by BOLTZMANN and the definition of entropy by CLAUSIUS, can be achieved by a proper choice of the entropy constant in Eq. (2.6) so that in thermodynamic equilibrium the constant occurring in Eq. (2.8) is zero:

$$H = S \quad (\text{thermodynamic equilibrium}). \quad (2.10)$$

For the following we define the average of some quantity Q by:

$$\langle Q \rangle = \frac{1}{n} \int Q F d\tau. \quad (2.11)$$

First we seek for a generalization of the concept of temperature. For a gas in thermodynamic equilibrium and at rest the temperature is related to the distribution function by the equation:

$$\frac{3}{2} n T = \frac{1}{2} m \langle \mathbf{v}^2 \rangle. \quad (2.12)$$

If the gas is in thermodynamic equilibrium but moving with a constant speed, $\langle \mathbf{v} \rangle$, that is if the distribution function is a displaced MAXWELLIAN distribution function, the relation between temperature and distribution function is given by:

$$\frac{3}{2} n T = \frac{1}{2} m [\langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2]. \quad (2.13)$$

The physical meaning of (2.13) is very clear. From the total kinetic energy $\frac{1}{2} m \langle \mathbf{v}^2 \rangle$ one has to subtract the kinetic energy resulting from a macroscopic ordered motion to obtain the disordered temperature motion.

We will use Eq. (2.13) to define the temperature for a velocity distribution which must not be in thermodynamic equilibrium, that is for any non-MAXWELLIAN velocity distribution. This generalized definition of temperature is from a physical point of view meaningful because the right-hand side of (2.13) is always equal to the disordered kinetic energy, which is not connected with a ordered macroscopic motion of the

gas. The expression in the square bracket on the right-hand side of Eq. (2.13) is called the mean square velocity fluctuation. One often writes for this quantity the abbreviation:

$$\langle (\Delta \mathbf{v})^2 \rangle = [\langle \mathbf{v}^2 \rangle - \langle \mathbf{v} \rangle^2]. \quad (2.14)$$

We thus may write for (2.13):

$$\frac{3}{2} n T = \frac{1}{2} m \langle (\Delta \mathbf{v})^2 \rangle. \quad (2.15)$$

Since the degree of disordered motion can be different in different directions the temperature should actually be defined by the dyadic tensor:

$$\frac{1}{2} n T_{ik} = \frac{1}{2} m \langle \Delta v_i \Delta v_k \rangle \quad (2.16)$$

where $\langle \Delta v_i \Delta v_k \rangle = \langle (v_i - \langle v_i \rangle)(v_k - \langle v_k \rangle) \rangle$. (2.17)

The "total temperature" T is then equal to one third of the trace of this tensor:

$$T = \frac{1}{3} T_i^i \quad (2.18)$$

and therefore:

$$\langle \Delta v_i \Delta v^i \rangle = \langle (\Delta \mathbf{v})^2 \rangle. \quad (2.19)$$

The specific heat for a monatomic gas is given by $3n\kappa/2$. Since the heat differential dQ must be a scalar the specific heat must be also a tensor of second rank. Because the specific heat cannot depend upon a direction it must be given by an isotropic tensor:

$$c_v^{ik} = \frac{1}{2} n \kappa \delta^{ik}. \quad (2.20)$$

In the expression of dQ we have to make for the change in internal energy $c_v dT$ the substitution:

$$\begin{aligned} c_v dT &= c_v^{ik} dT_{ik} \\ &= \frac{1}{2} n \kappa dT_{ik} \delta^{ik} = m n d \langle \Delta v_i \Delta v_k \rangle \delta^{ik} \\ &= \frac{3}{2} n \kappa dT = m n d \langle (\Delta \mathbf{v})^2 \rangle. \end{aligned} \quad (2.21)$$

The generalization of the pressure to non-equilibrium configurations is expressed with the same dyadic tensor defined by Eq. (2.17):

$$p_{ik} = m n \langle \Delta v_i \Delta v_k \rangle. \quad (2.22)$$

From (2.16) and (2.22) follows the generalized equation of state:

$$p_{ik} = n \kappa T_{ik}. \quad (2.23)$$

Taking the trace of (2.23):

$$p_i^i = n \kappa T_i^i \quad (2.24)$$

we obtain a relation between the "total pressure" $p = \frac{1}{3} p_i^i$ and the "total temperature" $T = \frac{1}{3} T_i^i$. This relation is the same as for an ideal gas:

$$p = n \kappa T. \quad (2.25)$$

For the work we have to put:

$$p dV = p_{ik} d\epsilon^{ik} \quad (2.26)$$

where ϵ^{ik} is the so-called deformation tensor³.

Inserting (2.26) and (2.21) into (2.9) results in:

$$dQ = m n \left[\frac{1}{2} \delta^{ik} d\langle \Delta v_i \Delta v_k \rangle + \langle \Delta v_i \Delta v_k \rangle d\epsilon^{ik} \right] \\ = m n \left[\frac{1}{2} d\langle (\Delta \mathbf{v})^2 \rangle + \langle \Delta v_i \Delta v_k \rangle d\epsilon^{ik} \right]. \quad (2.27)$$

From this we obtain an expression for the generalized macroscopic entropy defined by Eq. (2.6):

$$S - S_0 = 3 n \kappa \int \frac{\frac{1}{2} d\langle (\Delta \mathbf{v})^2 \rangle + \langle \Delta v_i \Delta v_k \rangle d\epsilon^{ik}}{\langle (\Delta \mathbf{v})^2 \rangle}. \quad (2.28)$$

In the case of LANDAU damping for electrostatic plasma waves, the volume between the initial and final state is unchanged. In order to compute the change in macroscopic entropy by LANDAU damping the second term in the numerator of the integrand in (2.28) vanishes. We thus get:

$$S - S_0 = \frac{3}{2} n \kappa \ln \left[\langle (\Delta \mathbf{v})^2 \rangle / \langle (\Delta \mathbf{v})^2 \rangle_0 \right]. \quad (2.29)$$

In considering LANDAU damping of electrostatic waves all quantities depend upon only one direction, the direction of wave propagation. For convenience sake, we may therefore consider as the only changing component of the temperature tensor the component along the direction of the wave propagation. If the wave propagates in the x direction the distribution function f then depends only upon the velocity component u in this direction. We thus may write for the temperature connected with this distribution function:

$$\frac{1}{2} \kappa (T - T_0) = \frac{1}{2} m \langle (\Delta u)^2 \rangle. \quad (2.30)$$

In Eq. (2.30) the quantity $\langle (\Delta u)^2 \rangle$ is defined as the mean square velocity fluctuation of the disturbance $f(u, t)$ rather than $F(u, t)$. We have brought the contribution of the mean square velocity fluctuation resulting from $f_0(u)$ to the left-hand side by observing that:

$$\frac{1}{2} \kappa T_0 = \frac{1}{2} (m/n) \int_{-\infty}^{+\infty} u^2 f_0(u) du. \quad (2.31)$$

Equation (2.31) expresses the equilibrium temperature T_0 by the equilibrium distribution at $t=0$. We have now:

$$\langle u^2 \rangle = (1/n) \int_{-\infty}^{+\infty} u^2 f(u, t) du, \\ \langle u \rangle = (1/n) \int_{-\infty}^{+\infty} u f(u, t) du. \quad (2.32)$$

From (1.7) and (2.7) the microscopic entropy according to BOLTZMANN is given by:

$$H = - (\kappa/n) \int_{-\infty}^{+\infty} (f_0 + f) \ln(f_0 + f) du \quad (2.33)$$

and the macroscopic entropy according to CLAUSIUS from (2.29) is given by:

$$S - S_0 = \frac{1}{2} n \kappa \ln \left[1 + \frac{m \langle (\Delta u)^2 \rangle}{\kappa T_0} \right]. \quad (2.34)$$

3. Landau Damping for Given Initial Conditions

For plane electrostatic waves, as already mentioned in the preceding paragraph, we can omit considering the temperature in all but one direction, that is the direction of wave propagation. If we choose the x axis along this direction and if u is the velocity component in this direction, we can write the linearized BOLTZMANN-VLASOV equation as follows:

$$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} - \frac{e}{m} \frac{\partial \varphi}{\partial x} \frac{df_0}{du} = 0. \quad (3.1)$$

For the equilibrium distribution $f_0(u)$ we assume a MAXWELL distribution:

$$f_0(u) = (n/\sqrt{2\pi}) (1/a \omega_p) \exp\{-u^2/2(a \omega_p)^2\} \quad (3.2)$$

where a is the DEBYE length:

$$a^2 = \kappa T / (4 \pi n e^2) \quad (3.3)$$

and ω_p the plasma frequency:

$$\omega_p^2 = 4 \pi n e^2 / m. \quad (3.4)$$

Eq. (1.9) can be written as:

$$\partial^2 \varphi / \partial x^2 = -4 \pi e \int_{-\infty}^{+\infty} f du. \quad (3.5)$$

In considering a wave propagating along the x direction we make the FOURIER transform:

$$f(u, x, t) = \sum_k f_k(u, t) e^{ikx} \quad (3.6)$$

and a similar transformation for φ . If we drop for convenience sake the index k we have for the FOURIER-transformed equation (3.1):

$$\partial f / \partial t + i k u f - i k (e/m) \varphi \cdot df_0/du = 0. \quad (3.7)$$

The electric field is expressed in terms of the potential φ by the equation:

$$\mathbf{E} = -\nabla \varphi \quad (3.8)$$

³ For instance: A. SOMMERFELD, *Mechanics of Deformable Bodies*, Academic Press, New York 1950, p. 4.

which becomes:

$$E_x = E = -\partial\varphi/\partial x. \quad (3.9)$$

The FOURIER-transformed Eq. (3.9) is:

$$E = -ik\varphi. \quad (3.10)$$

The FOURIER-transformed Eq. (3.5) is:

$$k^2\varphi = 4\pi e \int_{-\infty}^{+\infty} f_p(u) du. \quad (3.11)$$

We apply next on Eq. (3.7) and (3.11) a LAPLACE transform in time:

$$\begin{aligned} f_p(u) &= \int_0^{\infty} f(u, t) e^{-pt} dt, \\ f(u, t) &= (1/2\pi i) \int_{-i\infty+\varepsilon}^{+i\infty+\varepsilon} f_p(u) e^{pt} dp \end{aligned} \quad (3.12)$$

and similarly for φ . The result upon (3.7) and (3.11) is:

$$(p + ik) f_p - ik(e/m) \varphi_p \cdot df_0/du = g, \quad (3.13)$$

$$k^2 \varphi_p = 4\pi e \int_{-\infty}^{+\infty} f_p(u) du. \quad (3.14)$$

g contains the initial conditions necessary to formulate the initial value problem and is given by:

$$g(u) = f(u, 0). \quad (3.15)$$

$g(u)$ is thus the disturbance of the equilibrium distribution $f_0(u)$ at $t=0$.

From (3.13) we have:

$$f_p(u) = \frac{1}{p + ik} \left[g(u) + ik(e/m) \varphi_p \frac{df_0}{du} \right]. \quad (3.16)$$

Multiplying both sides of (3.16) with u , integrating over u from $-\infty$ to $+\infty$ and observing (3.14) we obtain for φ_p :

$$\varphi_p = \frac{4\pi e}{k^2} \frac{\int_{-\infty}^{+\infty} \frac{g(u)}{p + ik} du}{1 - \frac{4\pi i e^2}{k m} \int_{-\infty}^{+\infty} \frac{df_0}{du} \frac{du}{p + ik}}. \quad (3.17)$$

If φ_p is computed from (3.17) the solution can be inserted into (3.16) to obtain f_p . By inversion of the LAPLACE transform one then obtains finally $\varphi(x, t)$ and $f(u, t)$.

To make a simple choice for the initial conditions we consider a displaced MAXWELLIAN and expand it in a TAYLOR series:

$$f_0(u - u_0) = f_0(u) - u_0 \cdot df_0/du + \dots \quad (3.18)$$

Instead of a displaced MAXWELLIAN we choose as the initial conditions only the two first terms in its TAYLOR expansion. We thus have for the initial disturbance $g(u)$:

$$g(u) = -u_0 \cdot df_0/du. \quad (3.19)$$

Inserting (3.19) into (3.16) and (3.17) results in:

$$f_p(u) = \frac{1}{p + ik} [-u_0 + ik(e/m) \varphi_p] \frac{df_0}{du} \quad (3.20)$$

and

$$\varphi_p = -\frac{4\pi e}{k^2} \frac{u_0 \int_{-\infty}^{+\infty} \frac{df_0}{du} \frac{du}{p + ik}}{1 - \frac{4\pi i e^2}{k m} \int_{-\infty}^{+\infty} \frac{df_0}{du} \frac{du}{p + ik}}. \quad (3.21)$$

For the following it is convenient to introduce the abbreviation:

$$F(p) = \frac{4\pi i e^2}{m k} \int_{-\infty}^{+\infty} \frac{df_0}{du} \frac{du}{p + ik}. \quad (3.22)$$

With this we can write for (3.20):

$$f_p(u) = -u_0 \frac{df_0}{du} \frac{1}{p + ik} \frac{1}{1 - F(p)} \quad (3.23)$$

and for (3.21):

$$ik(e/m) \varphi_p = -u_0 \cdot F(p)/(1 - F(p)). \quad (3.24)$$

To study the behavior for $t \rightarrow \infty$ LANDAU continues $F(p)$, which as a LAPLACE transform is defined only for $\text{Re } p > 0$, analytically for $\text{Re } p < 0$ into the left half p plane by taking for the integral (3.22) a contour which keeps the pole at $u = ip/k$ always on the left side of the path of integration over u from $-\infty$ to $+\infty$ (see Fig. 2).

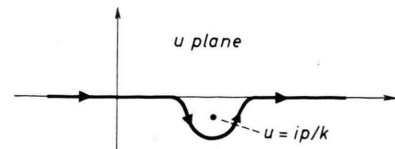


Fig. 2. Contour integration in the complex u -plane to evaluate integral (3.22) for $\text{Re } p < 0$.

If the pole at $u = ip/k$ is close to the real u axis which is the case for weak damping, LANDAU obtains for $F(p)$:

$$F(p) = -\frac{4\pi n e^2}{m p^2} + i \frac{4\pi^2 e^2}{m k^2} \frac{df_0(ip/k)}{du}; \quad \text{Re } p < 0. \quad (3.25)$$

From Eq. (3.24) it is clear that the asymptotic behavior of $\varphi(t)$ as $t \rightarrow \infty$ is determined by the poles of φ_p in the left half p plane. These poles are the roots of the equations:

$$1/F(p) = 0 \quad (3.26)$$

$$\text{and} \quad 1 - F(p) = 0. \quad (3.27)$$

From the expression of $F(p)$ given by (3.25) it follows that (3.26) has a root at $p=0$. The residue of this pole in the inverse LAPLACE transform is a constant. Since the potential φ is defined only to within a constant, the constant from the residue of the pole at $p=0$ can be omitted. For this reason, only the poles which are the roots of (3.27) determine the asymptotic behavior of $\varphi(t)$.

For the following it is convenient to consider instead of the roots of $1 - F(p) = 0$, the roots of the equation $p^2(1 - F(p)) = 0$. From Eq. (3.25) it follows that:

$$p^2(1 - F(p)) = p^2 + \omega_p^2 - \frac{i\pi\omega_p^2 p^2}{n k^2} \frac{df_0(i p/k)}{du}. \quad (3.28)$$

In the lowest approximation we neglect the last term on the right-hand side of Eq. (3.28) and have for the roots of $p^2(1 - F(p)) = 0$ the equation:

$$p_k^2 = -\omega_p^2 \quad (3.29)$$

with the two independent solutions:

$$p_k^\pm = \pm i \omega_p. \quad (3.30)$$

In the next higher approximation we substitute these values into the last term of the right-hand side of (3.28) and solve again for $p^2(1 - F(p)) = 0$.

The result is:

$$p_k^\pm = \pm i \omega_p \left[1 + \frac{i\pi\omega_p^2}{n k^2} \frac{df_0(\mp \omega_p/k)}{du} \right]^{\frac{1}{2}}, \quad (3.31)$$

or if we expand the square root in (3.31):

$$p_k^\pm = \pm i \omega_p \left[1 + \frac{i\pi\omega_p^2}{2 n k^2} \frac{df_0(\mp \omega_p/k)}{du} \right]. \quad (3.32)$$

With

$$\frac{df_0(u)}{du} = -\frac{n}{\sqrt{2}\pi} \frac{u}{(a\omega_p)^3} \cdot \exp\left\{-\frac{u^2}{2(a\omega_p)^2}\right\} \quad (3.33)$$

we obtain:

$$\frac{df_0(\mp \omega_p/k)}{du} = \pm \frac{n}{\sqrt{2}\pi} \frac{1}{k a^3 \omega_p^2} \cdot \exp\left\{-\frac{1}{2(k a)^2}\right\}. \quad (3.34)$$

Inserting (3.34) into (3.32) we then have:

$$p_k^\pm = \pm i \omega_p \left[1 \pm i \sqrt{\pi/8} (k a)^{-3} \cdot \exp\left\{\frac{1}{2} (k a)^{-2}\right\} \right]. \quad (3.35)$$

We can also write instead of (3.35):

$$p_k^\pm = \pm i \omega_p - \gamma, \quad (3.36)$$

where

$$\gamma = \sqrt{\pi/8} \omega_p (k a)^{-3} \cdot \exp\left\{\frac{1}{2} (k a)^{-2}\right\} \quad (3.37)$$

is the LANDAU damping constant. In Fig. 3 the location of the poles for $p^2(1 - F(p)) = 0$ are shown in the complex p plane.

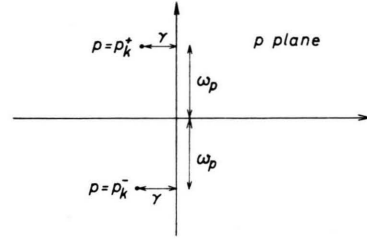


Fig. 3. Location of the poles of $p^2[1 - F(p)] = 0$ in the complex p -plane.

For $p^2(1 - F(p))$ we thus can write:

$$p^2(1 - F(p)) = (p - p_k^+)(p - p_k^-). \quad (3.38)$$

Or, since the pole at $p=0$ is of no significance, we we can write instead of (3.38) with sufficient accuracy:

$$(1 - F(p)) = -\omega_p^{-2}(p - p_k^+)(p - p_k^-). \quad (3.39)$$

For φ_p we then have according to (3.24):

$$\varphi_p = -\frac{i u_0 m \omega_p^2}{k e} \frac{1}{(p - p_k^+)(p - p_k^-)}. \quad (3.40)$$

We should remark that φ_p given by (3.40) is the analytic continuation of:

$$\varphi_p = \int_0^\infty \varphi(t) e^{-pt} dt \quad (3.41)$$

for $\text{Re } p < 0$.

$\varphi(t)$ is then obtained by the inverse LAPLACE transform:

$$\begin{aligned} \varphi(t) &= (1/2\pi i) \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \varphi_p e^{pt} dp \\ &= -\frac{i u_0 m \omega_p^2}{k e} \left[\frac{\exp(p_k^+ t)}{p_k^+ - p_k^-} + \frac{\exp(p_k^- t)}{p_k^- - p_k^+} \right]. \end{aligned} \quad (3.42)$$

But since

$$p_k^+ - p_k^- = 2 i \omega_p \quad (3.43)$$

we have:

$$\varphi(t) = -\frac{i u_0 m \omega_p}{k e} e^{-\gamma t} \sin \omega_p t. \quad (3.44)$$

To obtain from (3.44) the behavior of the electric field E we express E by φ with the help of Eq. (3.10). The result is:

$$E(t) = -(m/e) u_0 \omega_p e^{-\gamma t} \sin \omega_p t. \quad (3.45)$$

4. The Change in the Macroscopic Entropy

To compute the change in the macroscopic entropy, defined by Eq. (2.34), we have to compute the quantities $\langle u \rangle$ and $\langle u^2 \rangle$.

First we have:

$$\langle u \rangle_p = (1/n) \int_{-\infty}^{+\infty} u f_p du. \quad (4.1)$$

Inserting expression (3.23) into (4.1) results in:

$$\langle u \rangle_p = -\frac{1}{n} \frac{u_0}{1-F(p)} \int_{-\infty}^{+\infty} u \frac{df_0}{du} \frac{du}{p+iku}. \quad (4.2)$$

The integral

$$J = \int_{-\infty}^{+\infty} u \frac{df_0}{du} \frac{du}{p+iku} \quad (4.3)$$

must be analytically continued for $\text{Re } p < 0$. The integral (4.3) is continued analytically in just the same way as integral (3.22). The analytic continuation is therefore carried out in such a way that the path of integration is deformed into a contour for which the pole at $u = ip/k$ is always on the left side of the contour (see Fig. 4).

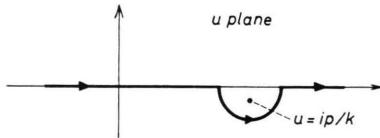


Fig. 4. Deformation of the path of integration of (4.3) for the analytic continuation into $\text{Re } p < 0$.

The integral can therefore be evaluated in a similar way as the integral (3.22). Considering the case for k small, we expand the denominator of the integrand in (4.3) in powers of iku/p , and find that the value of the integral to the second order of k is effectively the sum of two parts: the value when $k=0$ plus one half the residue of the pole at $u = ip/k$. We therefore have:

$$\begin{aligned} J &= \frac{1}{p} \int_{-\infty}^{+\infty} u \frac{df_0}{du} du + \pi i \text{Res} \left[u \frac{df_0}{du} \frac{1}{p+iku} \right]_{(u=ip/k)} \\ &= -\frac{n}{p} + \frac{\pi i p}{k^2} \frac{df_0(ip/k)}{du} \\ &= \frac{n}{p} \left[-1 + \frac{\pi i p}{n k^2} \frac{df_0(ip/k)}{du} \right]. \end{aligned} \quad (4.4)$$

One easily verifies that J can be expressed by $F(p)$ as follows:

$$J = n p F(p) / \omega_p^2. \quad (4.5)$$

We thus obtain for (4.2):

$$\langle u \rangle_p = -(u_0 / \omega_p^2) \cdot p F(p) / [1 - F(p)]. \quad (4.6)$$

Or using Eq. (3.39) we may also write:

$$\langle u \rangle_p = u_0 p F(p) / [(p - p_k^+) (p - p_k^-)]. \quad (4.7)$$

For the inverse LAPLACE transform again the poles with $p = p_k^\pm$ are the only ones of interest. We thus obtain:

$$\begin{aligned} \langle u \rangle &= (1/2 \pi i) \int_{-i\infty-\epsilon}^{+i\infty+\epsilon} \langle u \rangle_p e^{pt} dp \\ &= u_0 \left[\frac{p_k^+ \exp(p_k^+ t)}{p_k^+ - p_k^-} + \frac{p_k^- \exp(p_k^- t)}{p_k^- - p_k^+} \right]. \end{aligned} \quad (4.8)$$

We substitute for p_k^\pm , in the numerators of the fractions in the brackets, the lowest approximation, $p_k^\pm = \pm i \omega_p$, and have:

$$\langle u \rangle = u_0 e^{-\gamma t} \cos \omega_p t. \quad (4.9)$$

To compute $\langle u^2 \rangle$ we proceed in an indirect way making use of the fact that the energy of a particle moving in an electric field is conserved:

$$\frac{1}{2} m u^2 - e \varphi = [\frac{1}{2} m u^2 - e \varphi]_{t=0}. \quad (4.10)$$

The expression on the right-hand side of Eq. (4.10) is a constant of the motion. The constant shall be evaluated from the values of $\langle u^2 \rangle$ and φ at $t=0$. From (3.44) it follows that $\varphi(0) = 0$, therefore:

$$\frac{1}{2} m u^2 - e \varphi = [\frac{1}{2} m u^2]_{t=0}. \quad (4.11)$$

By expressing φ with Eq. (3.11) we can write for (4.11):

$$\frac{1}{2} m u^2 - \frac{4 \pi e^2}{k^2} \int_{-\infty}^{+\infty} f(u, t) du = [\frac{1}{2} m u^2]_{t=0}. \quad (4.12)$$

We multiply (4.12) on both sides by $f(u, t)$ and integrate over u from $-\infty$ to $+\infty$. We then obtain:

$$\begin{aligned} \frac{1}{2} m \langle u^2 \rangle &= -\frac{4 \pi e^2}{n k^2} \int_{-\infty}^{+\infty} f(u, t) du \cdot \int_{-\infty}^{+\infty} f(u, t) du \\ &= \frac{1}{2} m \langle u^2 \rangle_{t=0}. \end{aligned} \quad (4.13)$$

The product of the two integrals in (4.13) is the total electrostatic energy. From a well-known theorem of electrostatics the value of this product is just half the value of the square of the integral, omitting

the electrostatic selfenergy. Therefore:

$$\int_{-\infty}^{+\infty} f du \cdot \int_{-\infty}^{+\infty} f du \rightarrow \frac{1}{2} \left(\int_{-\infty}^{+\infty} f du \right)^2 \quad (4.14)$$

but since $\left(\int_{-\infty}^{+\infty} f(u, t) du \right)^2 = k^4 \varphi^2 / (4\pi e)^2$ (4.15)

we have $\frac{1}{2} m \langle u^2 \rangle - k^2 \varphi^2 / 8\pi n = \frac{1}{2} m \langle u^2 \rangle_{t=0}$ (4.16)

or $\frac{1}{2} m \langle u^2 \rangle + E^2 / 8\pi n = \frac{1}{2} m \langle u^2 \rangle_{t=0}$. (4.17)

From the generalized temperature defined by Eq. (2.30) we have:

$$\kappa T = \kappa T_0 + m \langle u^2 \rangle - m \langle u \rangle^2, \quad (4.18)$$

for $t=0$ we have in particular:

$$\kappa T_0 = [\kappa T]_{t=0} \quad (4.19)$$

therefore: $\langle u^2 \rangle_{t=0} = \langle u \rangle_{t=0}^2 = u_0^2$. (4.20)

From (4.17) follows therefore:

$$m \langle u^2 \rangle = m u_0^2 - E^2 / 4\pi n. \quad (4.21)$$

Or making use of (3.45):

$$m \langle u^2 \rangle = m u_0^2 (1 - e^{-2\gamma t} \sin^2 \omega_p t). \quad (4.22)$$

On the other hand from (4.9) it follows that:

$$m \langle u \rangle^2 = m u_0^2 e^{-2\gamma t} \cos^2 \omega_p t. \quad (4.23)$$

Inserting (4.22) and (4.23) into (4.18) we finally have:

$$\kappa T = \kappa T_0 + m u_0^2 (1 - e^{-2\gamma t}) \quad (4.24)$$

which becomes

$$T/T_0 = e^{2(S-S_0)/n\kappa} = 1 + (m u_0^2 / \kappa T_0) (1 - e^{-2\gamma t}). \quad (4.25)$$

At $t = \infty$ all the energy is converted into thermal energy:

$$\kappa T_\infty = \kappa T_0 + m u_0^2. \quad (4.26)$$

In Fig. 5 a plot is given for the change in the generalized temperature as a function of time.

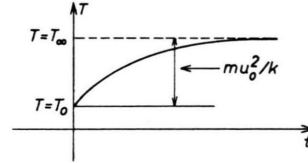


Fig. 5. Time-dependence of the generalized temperature for LANDAU damping of electrostatic plasma waves.

From Eq. (4.25) we conclude that for LANDAU damping the generalized temperature T is a steadily increasing function of time. The same applies to S , which is a satisfactory behavior for a quantity called entropy.

We would like to add the remark that for an initial disturbance in the distribution function which has the form of a δ -function $f(u, 0) = A \delta(u - u_0)$ one obtains the undamped VAN KAMPEN modes. Only in this special case $T = \text{const}$ and therefore also $S = \text{const}$.

Effects of Magnetic Shear on Density Gradient Drift Instabilities*

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The amounts of magnetic shear necessary to stabilize density gradient drift instabilities are estimated for a plasma with $1 \gg \beta > m/M$ where β = plasma pressure/magnetic pressure, m and M are the electron and ion masses. The stability criteria for electrostatic convective modes are shown to be slightly modified from those obtained previously for $\beta < m/M$. For $\beta > m/M$ however, we must also consider unstable ALFVÉN type modes. It is shown that the critical amount of magnetic shear is proportional to β^{-1} for the convective ALFVÉN type modes and to $\sqrt{\beta}$ for the non-convective modes.

§ 1. Introduction

Because of the necessity of stabilization of high temperature plasmas in thermonuclear devices, the

effects of magnetic shear on drift instabilities due to spatial inhomogeneity have recently been discussed extensively for both collisional and collisionless plasmas¹⁻⁷. In almost all of these discussions, espe-

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